Recall: Properties of determinants

A, B in $M_n(\mathbb{R})$.

1) $\det(AB) = \det(A)\det(B)$

2) If A is invertible,

$$\det(A^{-1}) = \frac{1}{\det(A)}$$

3) $\det(A^t) = \det(A)$

4) If B is obtained from A by multiplying a row or column of A by a constant c, then

$$\boxed{\det(B) = c\det(A)}$$

5) If B is obtained from A by interchanging two rows or columns, then $\boxed{\det(B) = -\det(A)}$

6) (new) If B is obtained from A by adding one row of A to another, then

$$\boxed{\det(A) = \det(B)}$$

<u>Example 1</u>: Suppose A, B are in $M_n(\mathbb{R})$. If $\det(A) = -2$ and $\det(B) = 5$, compute

$$\det(16A^4B^2).$$

Via property 1,

$$\det(16A^4B^2) = \det(16A^4)\det(B^2)$$

$$= \det(16A^4)\det(B)\det(B)$$

So

$$\det(16A^4B^2) = 25\det(16A^4)$$

$$= 25\det((2A)^4)$$

$$= 25(\det(2A))^4$$

again by property 1).

$$\det(2A) = \det((2I_n)A)$$

$$= \det(2I_n)\det(A)$$

$$= -2\det(2I_n)$$

$$= -2(2^n)$$

We get

$\det(2I_n) = 2^n$ by applying rule 4) n times, since we multiply each row by 2.

Finally,

$$\det(16A^4B^2) = 25(\det(2A))^4$$
$$= 25(-2 \cdot 2^n)^4$$
$$= \boxed{400 \cdot 2^{4n}}$$

Example 2:

Suppose $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

Compute

$$\det \begin{bmatrix} 3c & 3d \\ -4a & -4b \end{bmatrix}$$

if $\det(A) = 56$.

$$\det\begin{bmatrix} 3c & 3d \\ -4a & -4b \end{bmatrix}$$

$$= -\det\begin{bmatrix} -4a & -4b \\ 3c & 3d \end{bmatrix} \quad \text{(rule 5)}$$

$$= -3\det\begin{bmatrix} -4a & -4b \\ c & d \end{bmatrix} \quad \text{(rule 4)}$$

$$= (-3)(-4)\det\begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{(rule 4)}$$

$$= 12 \cdot 56 = \boxed{672}$$

Warning: $\det(A+B) \neq \det(A) + \det(B)$

in general

If $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$,

then $\det(A) = 1$, which

implies $\det(A^{-1}) = \frac{1}{1} = 1$.

So $\det(A) + \det(A^{-1}) = 2$.

But $A^{-1} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

Then

$$A + A^{-1} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

and so

$$\det(A + A^{-1}) = \boxed{0 \neq 2}$$

Special Cases

When is the determinant easy?

Upper/Lower Triangular Matrices

An $n \times n$ matrix $A = (a_{i,j})_{i,j=1}^{n}$

is called upper triangular

if $a_{i,j} = 0$ whenever $i > j$.

2x2

$$\begin{bmatrix} 3 & 5 \\ 0 & 6 \end{bmatrix}$$

3x3

$$\begin{bmatrix} -1 & 1 & 5 \\ 0 & 13 & 6 \\ 0 & 0 & -8 \end{bmatrix}$$

4x4

$$\begin{bmatrix} 16 & 12 & 0 & 0 \\ 0 & -2 & 1 & 22 \\ 0 & 0 & -8 & 4 \\ 0 & 0 & 0 & 13 \end{bmatrix}$$

A lower triangular matrix is the transpose of an upper triangular.

Trick: If $A = (a_{i,j})_{i,j=1}^{n}$ is upper triangular,

$$\boxed{\det(A) = \prod_{i=1}^{n} a_{i,i}}.$$

I.E. the determinant is the product of the diagonal entries.

Example 3:

Let $A = \begin{bmatrix} 16 & 12 & 0 & 0 \\ 0 & -2 & 1 & 22 \\ 0 & 0 & -8 & 4 \\ 0 & 0 & 0 & 13 \end{bmatrix}$.

Then $\det(A) = 16 \cdot (-2) \cdot (-8)(13)$

$$= (256)(13)$$

$$= \boxed{3328}$$

Example 4: Let A be in $M_n(\mathbb{R})$, suppose A is lower triangular with all diagonal entries equal to 56. Calculate $\det(A)$

$$\boxed{\det(A) = 56^n}$$

Note special best case!

A matrix that is both upper and lower triangular is called diagonal - the only (potentially) nonzero entries are the diagonal ones.

The determinant is again the product of diagonal entries.

Verifying Some of the Properties

Upper triangular

(2x2 case)

$$A = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}.$$

$$\det(A) = ad - b(0)$$

$$= ad$$

= product of diagonal entries.

General case – A is $n \times n$.

$$\det(A) = \sum_{\sigma \text{ permutation}} (-1)^{\operatorname{sign}(\sigma)} a_{1,\sigma(1)} a_{2,\sigma(2)} \cdots a_{n,\sigma(n)}$$

if $A = (a_{i,j})_{i,j=1}^{n}$.

Consider any permutation σ

and look at

$$a_{1,\sigma(1)} a_{2,\sigma(2)} \cdots a_{n,\sigma(n)}.$$

If $\sigma(m) = m$ for all $1 \leq m \leq n$,

then we get $a_{1,1} a_{2,2} \cdots a_{n,n}$.

If $\sigma(m) \neq m$ for some m, $1 \leq m \leq n$,

either $m > \sigma(m)$

or $m < \sigma(m)$.

If $m > \sigma(m)$, then since A is upper triangular,

$a_{m,\sigma(m)} = 0$ and so

$a_{1,\sigma(1)} \cdot a_{2,\sigma(2)} \cdots a_{n,\sigma(n)} = 0$.

If $m < \sigma(m)$,

set $k = \sigma(m)$. Then

$\sigma^{-1}(k) = m < \sigma(m) = k$

Then

$a_{k,\sigma^{-1}(k)} = 0$ but

not necessarily in our product! But there is an index j with $j > \sigma(j)$, again implies the product is zero.

Then

$$\boxed{\det(A) = a_{1,1} a_{2,2} \cdots a_{n,n}}$$